Journal homepage www.jzs.univsul.edu.iq
Journal of Zankoy Sulaimani
Part-A- (Pure and Applied Sciences)

Some Separation Axioms on $\omega_\delta -$ open set

Hardi N. Aziz [1,] Halgwrd Mohammed Darwesh[2], Adil K. Jabbar[2]

1 College of Education – Sulaimani university, Sulaimaniyah – Iraq
harde.aziz@univsul.edu.iq
2 College of Science – Sulaimani university, Sulaimaniyah – Iraq
halgwrd.darwesh@univsul.edu.iq , adil.jabbar@univsul.edu.iq

Article info

Original: 1 December 2018
Revised: 1 May 2019
Accepted: 21 May 2019
Published online: 20 June 2019

Key Words: $\omega_\delta -$ open set, $\omega_\delta - T_i, (i = 0,1,2), \omega_\delta -$ regular, $\omega_\delta -$ normal space.

Abstract

In this paper we introduce some types of separation axioms and study their basic relations with other types of separation axioms such as $\omega - T_i$ and $\delta - T_i$, for $i = 0,1,2$.We show that our separation axioms are coarser than $\omega - T_i$ and finer than $\delta - T_i$. We determine several crucial properties including that every locally countable space is $\omega_\delta - T_2$. Furthermore, we show that the notions T_2 and $\omega_\delta - T_2$ are coincide in anti – locally countable space.

Introduction

Let (X, τ) be a topological space, and $Int(-)$ and $Cl(-)$ be the usual interior and closure operators. Stone in [7] calls a subset A of X which satisfies $A = Int\big(Cl(A)\big)$ a regular open set. The collection of all regular open subsets of X forms a basis for a topology τ_s on X coarser than τ. The pair (X, τ_s) is called the semiregularization of (X, τ). In 1968, Velicko [8] defines the concept $\delta -$open and shows that the collection of all$\delta -$ open subsets of a topological space (X, τ) form topology on X, written τ_δ. Further, he shows that $\tau_s = \tau_\delta$ and $\tau_\delta \subseteq \tau$. Hdeib [2] introduces $\omega -$ closed sets and $\omega -$ open sets, see Definition 1.3. The collection of all $\omega -$ open subsets of X denoted by τ^ω is a topology on X, which is finer than τ. Halgwrd M. Darwesh [3] defines and studies a new type of sets called $\omega_\delta -$ open set. The $\omega_\delta -$ open set is coarser than $\omega -$ open set and it is independent of the concept of openness. Throughout this paper, the space $(R, \mathfrak{A})$ is Euclidean space on R, (R, T_{COC}) is co – countable topology on R and τ_{dis} is the discrete space.

Preliminaries

This section is devoted to provide the basic definitions and the most important results concerning $\omega -$ open set in topological space and mention some properties and characterizations needed in this work.

Definition 2.1. [8] A subset G of a space X is called $\delta -$ open if for each $x \in G$, there exists an open set U containing x such that $IntCl\, U \subseteq G$. For a subset A of a space X, the $Int_\delta(A), Cl_\delta(A)$ will be denoted the $\delta -$ interior and $\delta -$ closure of A, respectively.

Definition 2.2. [3] A subset U of a space X is called $\omega_\delta -$ open set if for each $x \in U$, there exists an open set G containing x such that $G - Int_\delta U$ is countable. The complement of $\omega_\delta -$ open set is called $\omega_\delta -$ closed sets. The family of all $\omega_\delta -$ open (resp., $\omega_\delta -$ closed) subsets of a space X are denoted by $\omega_\delta O(X)$ (resp., $\omega_\delta C(X)$).

Definition 2.3. [3] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be an $\omega_\delta -$ continuous function if the inverse image of each open subset of Y is an ω_δ-open subset of X.

Definition 2.4. [2] A subset U of a space X is called $\omega -$ open set if for each $x \in U$, there exists an open set G containing x such that $G - U$ is countable.

Definition 2.5. [2] A space X is said to be $\omega - T_1$ (resp. $\omega - T_2$) if for each pair of distinct points x and y of X, there exist $\omega -$ open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$ (resp. $U \cap V = \emptyset$).

Note that in [1], shown that every space is $\omega - T_1$ space.

Definition 2.6. [6] A space X is said to be anti-locally countable if each non-empty open subset of X is uncountable.

Theorem 2.7. [1] Let X be an anti-locally countable space. Then X is an $\omega - T_2$ space if and only if X is a $T_2 -$ space.

Definition 2.8. [6] A space X is said to be locally-countable if each point of X has a countable open neighborhood.

Proposition 2.9. [3] Let (X,τ) be a topological space. Then:

1) (X,τ) is locally countable if and only if $\omega_\delta O(X)$ is a discrete topology on X.

2) (X,τ) is anti – locally countable space if and only if $\left(X, \omega_\delta O(X)\right)$ is anti – locally countable.

Theorem 2.10. [3] Let U be a subset of a space X. Then U is $\omega_\delta -$ open if and only if for each $x \in U$, there exists an open set G containing x and a countable set C such that $(G - C) \subseteq Int_\delta(U)$.

Corollary 2.11. [1] Let (X,τ) be an anti-locally countable space and A be an $\omega -$ open subset of X. Then for each point $x \in A$, there exists an open subset G of X containing x such that $ClG \subseteq ClA$.

Definition 2.12. [4] A space X is called semi – regular if $\tau = \tau_\delta$.

Lemma 2.13. [6] If a space (X,τ) is anti-locally-countable, then:

1. $\omega ClA = ClA$, for every ω –open subset A of X.
2. $\omega IntA = IntA$, for every $\omega -$ closed subset A of X.

Definition 2.14. [10] A space X is said to be a Lindelof space if every open cover of X hasa countable sub-cover.

Definition 2.15. [5] A sequence of function $\{f_n\}, n = 1, 2, 3, \ldots$ is said to be uniformly convergent to f for a set E of values of x if for each $\epsilon > 0$, an integer N can be found such that $|f_n(x) - f(x)| < \epsilon$, for $n \geq N$ and all $x \in E$. A series $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by $\sum_{k=1}^{n} f_k(x) = S_n(x)$ converges uniformly on E.

Theorem 2.16. [10] Let $f_n: (Y,\tau) \to (R,\tau_U)$ be continuous functions for all $n \in N$, if $f: (Y,\tau) \to (R,\tau_U)$ is a function such that the series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent to $f(x)$. Then f is continuous.

The following definitions can be found in [9]:

Definition 2.17. A covering $\{B_\gamma: \gamma \in \Gamma\}$ is said to be a refinement of a covering $\{A_\lambda\}_{\lambda \in \Lambda}$ if for each $\gamma \in \Gamma$, there exists some $\lambda \in \Lambda$ such that $B_\gamma \subseteq A_\lambda$.

Definition 2.18. A family $\{A_\lambda\}_{\lambda \in \Lambda}$ of subsets of a space X is said to be point – finite if for each point x of X the set $\{\lambda \in \Lambda: x \in A_\lambda\}$ is finite.

Definition 2.19. A family $\{A_\lambda\}_{\lambda \in \Lambda}$ of subsets of a space X is said to be locally finite if for each point x of X there exists a neighborhood N_x of x such that the set $\{\lambda \in \Lambda: N_x \cap A_\lambda \neq \emptyset\}$ is finite.

Definition 2.20. A family $\{A_\lambda\}_{\lambda \in \Lambda}$ of subsets of a space X is said to be discrete if for each point x of X there exists a neighborhood N_x of x such that the set $\{\lambda \in \Lambda: N_x \cap A_\lambda \neq \emptyset\}$ has at most one member.

More Properties of $\omega_\delta -$ open sets

In this section we will prove more properties of $\omega_\delta -$ open sets and we prove the relations of $\omega_\delta -$ open sets with respect to the whole space and its subspace.

For any space X and $Y \subseteq X$, then $\left(\tau_{\omega_\delta}\right)_Y \neq (\tau_Y)_{\omega_\delta}$ in general, as shown in the following example:

Example 3.1. Consider (R, T_{COC}). Then $\tau_{\omega_\delta} = \{\emptyset, R\}$. Let $Y = \mathbb{Q}$. Then $(\tau_{\omega_\delta})_Y = \{\emptyset, \mathbb{Q}\}$. Since, $(\mathbb{Q}, \tau_{\mathbb{Q}})$ is locally countable, so by Proposition 2.9, $(\tau_{\mathbb{Q}})_{\omega_\delta} = \tau_{dis}$ in $\mathbb{Q}$. Hence, $(\tau_{\omega_\delta})_Y \neq (\tau_Y)_{\omega_\delta}$.

Proposition 3.2. If Y is an open subset in (X, τ), then $(\tau_{\omega_\delta})_Y \subseteq (\tau_Y)_{\omega_\delta}$.

Proof. Let $A \in (\tau_{\omega_\delta})_Y$ and $Y \in \tau$. Then there is $B \in \tau_{\omega_\delta}$ such that $A = B \cap Y$. For each $x \in A$, there exists an open set G in X and a countable set C such that $x \in G$ and $(G - C) \subseteq \delta Int(B)$. Since $x \in (G \cap Y) \in \tau_Y$, so $G \cap (Y - C) = (G - C) \cap Y \subseteq (\delta Int(B) \cap Y) \subseteq \delta Int_Y(A)$. Hence, $A \in (\tau_Y)_{\omega_\delta}$.

Corollary 3.3. If Y is an open subset in (X, τ) and $A \in (\tau_Y)_{\omega_\delta}$ such that $A \subseteq \delta Int(Y)$, then $A \in \tau_{\omega_\delta}$.

Theorem 3.4. Let Y be a $\delta -$ open subset of X. If A is $\omega_\delta -$ open set in Y, then A is $\omega_\delta -$ open set in X.

Proof. Let A be an $\omega_\delta -$ open in Y. Then for each $x \in A$, there exists an open set G in Y and a countable set C such that $x \in G$ and $(G - C) \subseteq \delta Int_Y(A)$. Since G is open in Y and Y is open in X, so G is open in X. Since, $\delta Int_Y(A) = \delta Int_Y(A) \cap Y = \delta Int(A) \cap \delta Int(Y) = \delta Int(A)$, so $(G - C) \subseteq \delta Int(A)$. Thus, by **Theorem 2.10**, A is $\omega_\delta -$ open in X.

Theorem 3.5. If Y is a $\delta -$ open subspace of (X, τ), then $(\tau_{\omega_\delta})_Y = (\tau_Y)_{\omega_\delta}$.

Proof. Let $A \in (\tau_{\omega_\delta})_Y$. Then $A = B \cap Y$, for some $B \in \tau_{\omega_\delta}$ and for each $x \in A$, there exists an open set G in X and a countable set C such that $(G - C) \subseteq \delta Int(B)$, then $G \cap Y$ is open in Y, so $x \in (G \cap Y)$ and $G \cap (Y - C) = (G - C) \cap Y \subseteq (\delta Int(B) \cap Y) = \delta Int(B) \cap \delta Int(Y) = \delta Int(B \cap Y) = \delta Int(A) \subseteq \delta Int_Y(A)$, so by Theorem 2.10, $A \in (\tau_Y)_{\omega_\delta}$.

Conversely, let $A \in (\tau_Y)_{\omega_\delta}$. Then for each $x \in A$, there is $G \in \tau_Y$ such that $x \in G$ and a countable set C such that $(G - C) \subseteq \delta Int_Y(A)$. Since G is open in X and in view of Theorem 3.4, $\delta Int_Y(A) = \delta Int(A)$, so $G - C \subseteq \delta Int(A)$. Hence, $A \in \tau_{\omega_\delta}$, so $A = (A \cap Y) \in (\tau_{\omega_\delta})_Y$. Therefore, $(\tau_{\omega_\delta})_Y = (\tau_Y)_{\omega_\delta}$.

Proposition 3.6. If Y is an $\omega_\delta -$ open subset of a spac (X, τ), then $(\tau_{\omega_\delta})_Y \subseteq (\tau_Y)_{\omega_\delta}$.

Proof. Let $A = B \cap Y$, where $B \in \tau_{\omega_\delta}$. Then, for each $x \in A$, there are two open sets G_1, G_2 and two countable sets C_1, C_2 such that $x \in (G_1 \cap G_2), (G_1 - C_1) \subseteq \delta Int(B)$ and $(G_2 - C_2) \subseteq \delta Int(Y)$. Since $U = (G_1 \cap G_2) \cap Y \in \tau_Y, x \in U$ and $C = C_1 \cup C_2$ is a countable set, then $(U - C) \subseteq (G_1 \cap G_2) - C \subseteq ((G_1 - C_1) \cap (G_2 - C_2)) \subseteq (\delta Int(B) \cap \delta Int(Y)) = \delta Int(B \cap Y) = \delta Int(A) \subseteq \delta Int_Y(A)$. So $A \in (\tau_Y)_{\omega_\delta}$.

Lemma 3.7: If a space (X, τ) is anti – locally countable, then:

1) $Cl(A) \subseteq \omega_\delta Cl(A)$, for every $\omega -$ open subset A of X.

2) $\omega_\delta Int(A) \subseteq Int(A)$, for every $\omega -$ open subset A of X.

Proof. 1) Let A be any $\omega -$ open subset of X. Then, by Lemma 2.13, $Cl(A) = \omega Cl(A)$. Since, $\omega Cl(A) \subseteq \omega_\delta Cl(A)$, so $Cl(A) \subseteq \omega_\delta Cl(A)$.

2) Analogous to part (1).

Corollary 3.8. Let X be an anti – locally countable space and A be an $\omega_\delta -$ open subset of a space X. Then for each $x \in A$, there exists an open set G containing x such that $Cl(G) \subseteq Cl(A)$.

Proof. It follows from Corollary 2.11.

Proposition 3.9. Let A be a subset of a $\delta -$open subspace (Y, τ_Y) of a space (X, τ). Then:

1) $\omega_\delta Cl_Y(A) = \omega_\delta Cl(A) \cap Y$.

2) A is $\omega_\delta -$ closed in Y if and only if $A = F \cap Y$, for some $\omega_\delta -$ closed subset F of X.

Proposition 3.10. Let $A \subseteq Y$ and (Y, τ_Y) be a subspace of (X, τ). Then:

1) If A is $\omega_\delta -$ open in Y and Y is $\delta -$ open in X, then A is $\omega_\delta -$ open in X.

2) If A is $\omega_\delta -$ open in X and Y is $\omega_\delta -$ open in X, then A is $\omega_\delta -$ open in Y.

3) If Y is $\delta -$ open in X, then A is $\omega_\delta -$ open in Y if and only if it is $\omega_\delta -$ open in X.

4) If Y is $\delta -$ open, then A is $\omega_\delta -$ closed in Y if and only if it is $\omega_\delta -$ closed in X.

Proof.

1) Follows from Theorem 3.4.

2) Let A, Y be $\omega_\delta -$ open subsets in X. Then by Proposition 3.6, $\mathrm{A} = \mathrm{A} \cap \mathrm{Y} \in \left(\tau_{\omega_\delta}\right)_{\mathrm{Y}} \subseteq (\tau_{\mathrm{Y}})_{\omega_\delta}$. So A is $\omega_\delta -$ open in Y.

3) Follows from part (1) and part (2).

4) Let Y be $\delta -$ open. Then, if A is $\omega_\delta -$ closed in Yand since $\mathrm{A} = \omega_\delta\mathrm{Cl}_{\mathrm{Y}}(\mathrm{A}) = \omega_\delta\mathrm{Cl}(\mathrm{A}) \cap \mathrm{Y} = \omega_\delta\mathrm{Cl}(\mathrm{A})$. So it is $\omega_\delta -$ closed in X. If A is $\omega_\delta -$ closed in X, then $\omega_\delta\mathrm{Cl}(\mathrm{A}) = \mathrm{A}$, so $\omega_\delta\mathrm{Cl}(\mathrm{A}) \cap \mathrm{Y} = \mathrm{A} \cap \mathrm{Y} = \mathrm{A}$. By Proposition 2.58, $\mathrm{A} = \omega_\delta\mathrm{Cl}_{\mathrm{Y}}(\mathrm{A})$, so by [Lemma 3.16, 3], A is $\omega_\delta -$ closed in Y.

Lemma 3.11 [9]. A closed subset of Lindelof space is Lindelof.

Proposition 3.12. For any closed subset A of a lindelof space X. If A is $\omega_\delta -$ open, then $A - \delta Int(A)$ is countable set.

Proof. Since A is $\omega_\delta -$ open, then there is an open set G_x and a countable set C_x such that $x \in G_x$ and $(G_x - \delta Int(A)) \subseteq C_x$, for each $x \in A$. Thus, $\{G_x : x \in A\}$ is an open cover of A, so by Lemma 3.11**,** A is lindelof in X. So there is a countable subset A_0 of A such that $A \subseteq \bigcup_{x \in A_0} G_x$. Then$(A - \delta Int(A)) \subseteq \bigcup_{x \in A_0}(G_x - \delta Int(A)) = \bigcup_{x \in A_0}\left(G_x - \delta Int(A)\right) \subseteq \bigcup_{x \in A_0} C_x$, which is countable. Hence, $A - \delta Int(A)$ is countable.

Proposition 3.13. A closed subset of a Lindelof space X is $\omega_\delta -$ open if and only if it is $\omega -$ open.

Proof. Clearly, every $\omega_\delta -$ open set is $\omega -$ open, so we have only top show if A is $\omega -$ open, then it is $\omega_\delta -$ open. For this, let A be an $\omega -$ open subset of X. For each $x \in A$, there is an open set G and a countable set C such that $G - C \subseteq A$, so $G - C - \left(\delta Int(A)\right) \subseteq A - \left(A - \delta Int(A)\right) = \delta Int(A)$. But From Proposition 3.12, we have $A - \delta Int(A)$ is countable. Therefore, A is $\omega_\delta -$ open.

$\omega_\delta - T_i, (i = 0, 1, 2)$ spaces and some properties of $\omega_\delta -$ continuous Functions

The main purpose of this section is to define $\omega_\delta - T_i, (i - 0,1,2)$ spaces, and obtain some characterizations, properties, and relationships. We start with the following definition:

Definition 4.1. A space X is said to be

1- $\omega_\delta - T_0$ space if for each pair of distinct points x and y of X, there exists an $\omega_\delta -$ open set U containing one of them, but not the other.

2- $\omega_\delta - T_1$ space if for each pair of distinct points of X, there exist $\omega_\delta -$ open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $y \notin V$.

3- $\omega_\delta - T_2$ space if for each distinct points x and y of X, there exists disjoint $\omega_\delta -$ open sets U and V containing x and y, respectively.

Theorem 4.2. Let $f : X \to Y$ be a bijection $\omega_\delta -$ continuous function and Y is a $T_0 -$ space. Then X is an $\omega_\delta - T_0$ space.

Theorem 4.3. A space X is an $\omega_\delta - T_0$ space if and only if $\omega_\delta Cl(\{x\}) \neq \omega_\delta Cl(\{y\})$, for each pair of distinct points x and y of X.

Lemma 4.4. Every locally countable space is $\omega_\delta - T_2$ space.

Proof. Follows from Proposition 2.9.

Definition 4.5: A function $f : (X, \tau) \to (Y, \sigma)$ is called pre $\omega_\delta -$ open function if and only if f sends $\omega_\delta -$open sets of X into $\omega_\delta -$ open sets of Y.

Definition 4.6. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be pre $\omega_\delta -$ closed if the image of each $\omega_\delta -$ closed set of X is $\omega_\delta -$ closed in Y.

The pre $\omega_\delta -$ open and open function are independent, as shown in the following examples:

Example 4.7. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $Y = \{p, q, r\}$, $\tau' = \{\emptyset, \{r\}, \{p, q\}, Y\}$. Then a function $f : X \to Y$ defined by $f(a) = p, f(b) = q$ and $f(c) = p$ is pre $\omega_\delta -$ open function, but not open function, since $f(\{a\}) = \{p\}$ which is not $\tau' -$ open set.

Example 4.8. Consider the topologies $\tau = \{\emptyset, A, \mathbb{R} - A, \mathbb{R}\}, A = (0,1)$ and $\sigma = \{\emptyset, A, \{10\}, A \cup \{10\}, (0,10], \mathbb{R}\}$ of the set of all real numbers $\mathbb{R}$.

Let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ be a function given by $f(x) = \begin{cases} x, & x \in A \\ 10, & x \notin A \end{cases}$.

Then f is open but not pre $\omega_\delta -$ open function.

Proposition 4.9. If $f: (X, \tau) \to (Y, \sigma)$ is an onto, open function and X is locally countable, then f is pre ω_δ −open.

Proposition 4.10. If $f: (X, \tau) \to (Y, \sigma)$ is an open function and Y is semi – regular space, then f is pre $\omega_\delta -$ open.

Corollary 4.11. If $f: (X, \tau) \to (Y, \sigma)$ is an open function and Y is regular space, then f is pre $\omega_\delta -$ open.

Theorem 4.12. Every homeomorphism function maps $\omega_\delta -$ open sets onto $\omega_\delta -$ open sets.

Proof. Let $f: X \to Y$ be a homeomorphism function and let G be an $\omega_\delta -$ open subset of X. To show $f(G)$ is $\omega_\delta -$ open. Let $y \in f(G)$. Then there exists $x \in G$ such that $f(x) = y$, since f is bijective. But G is $\omega_\delta -$ open set, so there exists an open set V such that $x \in V$ and $V - Int_\delta(G) = C$ is countable. Since $f(V)$ is open in Y, since, f is open, $y \in f(V)$ and $f(C) = f\big(V - Int_\delta(G)\big) = f(V) - f\big(Int_\delta(G)\big) = f(V) - Int_\delta\big(f(G)\big)$. So, $f(V) - Int_\delta\big(f(G)\big)$ is countable. Thus, $f(G)$ is $\omega_\delta -$ open set.

Proposition 4.13. The property of a space being $\omega_\delta - T_i$ space $(i = 0,1,2)$ is a topological property.

From the fact that, every $\delta -$ open set is $\omega_\delta -$ open set, we obtain that every $\delta - T_i$ space is $\omega_\delta - T_i$, where $i = 0, 1, 2$.

The converse is not true in general, as shown in the following example:

Example 4.14. Let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then, $\tau_\delta = \{\emptyset, X\}$ and by Proposition 2.9, $\omega_\delta O(X) = \tau_{dis}$. Hence, (X, τ) is $\omega_\delta - T_i$, but it is not $\delta - T_i$, for each $i = 0, 1, 2$.

Example 4.15. Let X be uncountable set, which is equipped with the co – countable topology. Then $\tau_\omega = \tau$, but $\tau_{\omega_\delta} = \{\emptyset, X\}$, so (X, τ) is both T_i and $\omega - T_i$, $(i = 0, 1)$, but it is not $\omega_\delta - T_i$, $(i = 0, 1, 2)$.

Lemma 4.16. Every T_2 space is an $\omega_\delta - T_2$ space.

Proof. Let $x \neq y$ in a $T_2 -$ space X. Then, there are two open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$. So $IntCl(U) \cap IntCl(V) = \emptyset$. Since $x \in U \subseteq IntCl(U) \subseteq \tau_\delta \subseteq \tau_{\omega_\delta}$ and $y \in V \subseteq IntCl(V) \subseteq \tau_\delta \subseteq \tau_{\omega_\delta}$. So X is an $\omega_\delta - T_2$ space.

The converse of Lemma 4.16 is not true in general, as shown in the following example:

Example 4.17. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $\omega_\delta O(X) = \tau_{dis}$. Hence, (X, τ) is $\omega_\delta - T_2$, but it is not $T_2 -$ space.

Theorem 4.18. Let X be an anti – locally countable space. Then X is an $\omega_\delta - T_2$ space if and only if it is a $T_2 -$ space.

Proof. Let X be an anti – locally countable $\omega_\delta - T_2$ space. Then X is an $\omega - T_2$ space, so by Theorem 2.7, X is a $T_2 -$ space. The converse part is follows from Lemma 4.16.

Proposition 4.19. Let (X, τ) be a space. Then the following are equivalent:

1- (X, τ) is $\omega_\delta - T_0$.

2- Each subspace of $(X, \tau_{\omega_\delta})$ is a $T_0 -$ space.

3- Each proper subspace of $(X, \tau_{\omega_\delta})$ is a $T_0 -$ space.

Proof.

1 $\Longrightarrow$ **2**: Let (X, τ) be an $\omega_\delta - T_0$ space. To show $\left(Y, (\tau_{\omega_\delta})_Y\right)$ is a T_0 space. We take $x \neq y$ in Y, then there exists an $\omega_\delta -$ open set G in X contains one of them, but not the other, say $x \in G$ but $y \notin G$, so $x \in (G \cap Y) \in (\tau_{\omega_\delta})_Y$ and $y \notin (G \cap Y)$. Hence $\left(Y, (\tau_{\omega_\delta})_Y\right)$ is a $T_0 -$ space.

2 $\Longrightarrow$ **3**: Obvious.

3 $\Longrightarrow$ **1**: Let $\left(Y, (\tau_{\omega_\delta})_Y\right)$ be a $T_0 -$ space, for each proper subset Y in X. If X is a countable set, then by **Proposition 2.9**, $\tau_{\omega_\delta} = \tau_{dis}$, so (X, τ) is an $\omega_\delta - T_0$ space. Now, we suppose that X is an uncountable set. Since $Y = \{x, y\}$ is a proper subset of X, so by our hypothesis, there exists an $\omega_\delta -$ open set G in X such that the set $V = G \cap Y$ contains x, but not y. So the set G contains x, but not y. Hence, X is an $\omega_\delta - T_0$ space.

Remark 4.20. Note that in Proposition 4.19**,** if we replace $\omega_\delta - T_0$ by T_0, part (3) does not imply part (1), for example, if $X = \{a, b\}, \tau = \{\emptyset, X\}$, then (X, τ) is not T_0 but each nonempty proper subset of (X, τ) is T_0.

Example 4.21. Consider (R, T_{COC}) is a space, which is $\omega - T_i, (i = 0, 1)$, but is not $\omega_\delta - T_i, (i = 0, 1)$ space.

$\boldsymbol{\omega_\delta}$ − regular and $\boldsymbol{\omega_\delta}$ − normal spaces

In this section, we introduce the concepts of $\omega_\delta -$ regularity, $\omega_\delta -$ normality, and we give some of their characterizations and basic properties.

Definition 5.1. A space (X, τ) is called $\omega_\delta -$ regular space if each $\omega_\delta -$ closed subset H of X and a point x in X such that $x \notin H$, there exist disjoint $\omega_\delta -$ open sets U and V containing x and H, respectively.

$\omega_\delta -$ regular and semi – regular are independent, as shown in the following two examples:

Example 5.2. Consider the usual space $(\mathbb{R}, \mathfrak{A})$ is semi – regular which is not $\omega_\delta -$ regular.

Example 5.3. Let $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$. Then (X, τ) is an $\omega_\delta -$ regular, which is not semi – regular.

Theorem 5.4. A space X is an $\omega_\delta -$ regular if and only if for each point $x \in X$ and each $\omega_\delta -$ open set G containing x, there exists an $\omega_\delta -$ open set V such that $x \in V \subseteq \omega_\delta Cl(V) \subseteq G$.

From the following examples, we see that regular and $\omega_\delta -$ regular are independent:

Example 5.5. Let $X = \{a, b\}, \tau = \{\emptyset, \{a\}, X\}$. Then $\omega_\delta O(X) = \tau_{dis}$.
Hence, (X, τ) is $\omega_\delta -$ regular, but not regular.

Example 5.6. Clearly $(\mathbb{R}, \mathfrak{A})$ is regular, but it is not $\omega_\delta -$ regular space.

Corollary 5.7. Every locally countable space is $\omega_\delta -$ regular.

Proof. Follows from Proposition 2.9.

Corollary 5.8. Every $\omega_\delta -$ regular $\omega_\delta - T_1$ space is an $\omega_\delta - T_2$ space.

Theorem 5.9. The property of being a space is $\omega_\delta -$ regular is topological property.

Theorem 5.10. Let $f_n: (X, \tau) \to (R, \mathfrak{A})$ be $\omega_\delta -$ continuous functions for all $n \in N$, if $f: (X, \tau) \to (R, \mathfrak{A})$ is a function such that the series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent to $f(x)$. Then f is $\omega_\delta -$ continuous.

Proof.
Since for each $\mathrm{n} \in \mathrm{N}, \mathrm{f_n}: (\mathrm{X}, \tau) \to (\mathrm{R}, \tau_\mathrm{U})$ are $\omega_\delta -$ continuous functions, then by [Theorem 4.2, 3] $\mathrm{f_n}: \left(\mathrm{X}, \omega_\delta \mathrm{O}(\mathrm{X})\right) \to (\mathrm{R}, \tau_\mathrm{U})$ are continuous functions, for all $\mathrm{n} \in \mathrm{N}$. So by Theorem 2.16**,** $\mathrm{f}: \left(\mathrm{X}, \omega_\delta \mathrm{O}(\mathrm{X})\right) \to (\mathrm{R}, \tau_\mathrm{U})$ is continuous. Hence, by [Theorem 4.2, 3] $\mathrm{f}: (\mathrm{X}, \tau) \to (\mathrm{R}, \tau_\mathrm{U})$ is $\omega_\delta -$ continuous.

Definition 5.11. A space X is said to be $\omega_\delta -$ normal space if for each pair of disjoint $\omega_\delta -$ closed sets A and B in X, there exist disjoint $\omega_\delta -$ open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 5.12. A space X is $\omega_\delta -$ normal if and only if whenever G is $\omega_\delta -$ open, A is $\omega_\delta -$ closed and $A \subseteq G$, then there exists an $\omega_\delta -$ open set U with $A \subseteq U \subseteq \omega_\delta Cl(U) \subseteq G$.

Proposition 5.13. If Y is an $\omega_\delta -$ closed subset of an $\omega_\delta -$ normal space (X, τ), then $\left(Y, \left(\tau_{\omega_\delta}\right)_Y\right)$ is normal.

Theorem 5.14. Let $f: (X, \tau) \to (Y, \sigma)$ be an onto $\omega_\delta -$ continuous function, which maps $\omega_\delta -$ closed sets onto closed sets. If (X, τ) is $\omega_\delta -$ normal, then Y is normal.

Proof. Let A, B be two disjoint closed subsets of Y. Since, f is $\omega_\delta -$ continuous, so $f^{-1}(A), f^{-1}(B)$ are disjoint $\omega_\delta -$ closed subsets of X. Since, X is $\omega_\delta -$ normal, then there are disjoint $\omega_\delta -$ open sets U and V in X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. By hypothesis, we have $f(X - U)$ and $f(X - V)$ are closed sets in Y, so $U' = Y - f(X - U), V' = Y - f(X - V)$ are open sets in X. Since, $(X - U) \cup (X - V) = X$, so $f(X - U) \cup f(X - V) = Y$. Thus, $U' \cap V' = \emptyset$. Now, if $y \in A$, then $f^{-1}(\{y\}) \subseteq f^{-1}(A) \subseteq U$, so $f^{-1}(\{y\}) \cap (X - U) = \emptyset$. That is, $y \notin f(X - U)$. Therefore, $y \in Y - f(X - U) = U'$. Thus $A \subseteq U'$. Similarly, $B \subseteq V'$. Hence, Y is normal space.

Theorem 5.15. For any space X, the following statements are equivalent:

1) X is $\omega_\delta -$ normal.

2) For each nonempty $\omega_\delta -$ closed subset A and $\omega_\delta -$ open subset B of X such that $A \subseteq B$, there exists an $\omega_\delta -$ continuous function $f: X \to I$ such that $f(A) = \{0\}$ and $f(X - B) = \{1\}$, where $I = [0,1]$.

3) For each pair of nonempty disjoint $\omega_\delta -$ closed subsets F and H of X, there exists an $\omega_\delta -$ continuous function $f: X \to I$ such that $f(F) = \{0\}$ and $f(H) = \{1\}$.

Proof.

1 $\Longrightarrow$ 2: Suppose that B is $\omega_\delta -$ closed subset of an $\omega_\delta -$ normal space X, which containing a nonempty $\omega_\delta -$ closed A of X. Then, by Theorem 5.12, there exists an $\omega_\delta -$ open set, which we denoted by $U_{\frac{1}{2}}$ such that $A \subseteq U_{\frac{1}{2}} \subseteq \omega_\delta Cl\left(U_{\frac{1}{2}}\right) \subseteq B$. Then, $U_{\frac{1}{2}}$ and B are $\omega_\delta -$ open subsets of X containing the $\omega_\delta -$ closed sets A and $\omega_\delta Cl\left(U_{\frac{1}{2}}\right)$, respectively. In the same way, there exist an $\omega_\delta -$ open sets, say $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that $A \subseteq U_{\frac{1}{4}} \subseteq \omega_\delta Cl\left(U_{\frac{1}{4}}\right) \subseteq U_{\frac{1}{2}}$ and $\omega_\delta Cl\left(U_{\frac{1}{2}}\right) \subseteq U_{\frac{3}{4}} \subseteq \omega_\delta Cl\left(U_{\frac{3}{4}}\right) \subseteq B$. By containing this process, for each rational numbers in the open interval $(0,1)$ of the form $\lambda = \frac{m}{2^n}$, where $n = 1, 2, \dots$ and $m = 1, 2, \dots, 2^{n-1}$, we obtain $\omega_\delta -$ open sets of the form U_λ such that for each $t < \lambda$, then $A \subseteq U_t \subseteq \omega_\delta Cl(U_t) \subseteq U_\lambda \subseteq \omega_\delta Cl(U_\lambda)$. We denote the set of all such rational numbers by ψ, and define
$f: X \to I$ as follows:

$$f(x) = \begin{cases} 1 & if\ x \in X - B \\ \inf\{\lambda: \lambda \in \psi\ and\ x \in U_\lambda\} \end{cases}$$

$f(X - B) = \{1\}$, and if $x \in A$, then $x \in U_\lambda$ for all $\lambda \in \psi$. Therefore, by definition of f, we have $f(x) = \inf(\psi) = 0$. Hence, $f(B) = \{0\}$ and $f(x) \in I$, for all $x \in X$. To show f is an $\omega_\delta -$ continuous function. Since, the intervals of the form $[0, a)$ and $(b, 1]$, where $a, b \in (0,1)$ form an open subbase of the space I. If $x \in U_\lambda$, for some $\lambda < a$, then $f(x) = \inf\{t: t \in \psi, for\ all\ x \in U_t\} = r \leq \lambda < a$. Thus, $0 \leq f(x) < a$. If $f(x) = 0$, then $x \in U_\lambda$, for all $\lambda \in \psi$. Hence, $x \in U_\lambda$, for some $\lambda < a$. If $0 < f(x) < a$, by definition of f, we have $f(x) = \{t: t \in \psi\ and\ x \in U_t\} < a$, since, $a < 1$. Thus, $f(x) = \lambda$, for some $\lambda < a$ and hence $x \in U_\lambda$, for some $\lambda < a$. Therefore, we conclude that $0 \leq f(x) < a$ if and only if $x \in U_\lambda$, for some $\lambda < a$. Hence, $f^{-1}\big([0, a)\big) = \cup\, \{U_\lambda: \lambda \in \psi\ and\ x \in U_\lambda\}$, which is an $\omega_\delta -$ open subset of X. Also, we assert that, $0 \leq f(x) \leq b$ if and only if $x \in U_\lambda$, for all $\lambda > b$. Let $x \in X$ such that $0 \leq f(x) \leq b$. It is evident that $f(x) < \lambda$, for all $\lambda > b$, which implies that $x \in U_\lambda$, for all $\lambda > b$. For the converse, let $x \in U_\lambda$, for all $\lambda > b$. Then $f(x) \leq \lambda$, for all $\lambda > b$. Thus $f(x) \leq b$ and from definition of f that $f(x) \geq 0$. This prove our assertion.

Since, for all $\lambda > b$, there is $r \in \psi$ such that $\lambda > r > b$. Then, $\omega_\delta Cl(U_r) \subseteq U_\lambda$. Consequently, we have $\cap\, \{U_\lambda: \lambda \in \psi\ and\ \lambda > b\} = \cap\, \{\omega_\delta Cl(U_r): r \in \psi\ and\ r > b\}$. Therefore, $f^{-1}([0, b]) = \{x: 0 \leq f(x) \leq b\} = \cap\, \{U_\lambda: \lambda \in \psi\ and\ \lambda > b\} = \cap\, \{\omega_\delta Cl(U_r): r \in \psi\ and\ r > b\}$. Since, $f^{-1}([b, 1]) = f^{-1}(I - [0, b]) = X - f^{-1}([0, b]) = \cup\, \{X - \omega_\delta Cl(U_r): r \in \psi\ and\ r > b\}$, which is an $\omega_\delta -$ open. Thus, f is an $\omega_\delta -$ continuous function.

2 $\Longrightarrow$ 3: Clear.

3 $\Longrightarrow$ 1: Let A and B be two disjoint $\omega_\delta -$ closed subsets of X. Then by hypothesis, there exists an $\omega_\delta -$ continuous function $f: X \to I$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then the disjoint open sets $\left[\frac{0,1}{2}\right)$ and $\left(\frac{1}{2}, 1\right]$ in I containing $f(A)$ and $f(B)$, respectively. The $\omega_\delta -$ continuity of f gives that $f^{-1}\left(\left[\frac{0,1}{2}\right)\right)$ and $f^{-1}\left(\left(\frac{1}{2}, 1\right]\right)$ are disjoint $\omega_\delta -$ open sets in X containing A and B, respectively. Hence, X is $\omega_\delta -$ normal.

Corollary 5.16. Every $\omega_\delta -$ normal $\omega_\delta - T_1$ space is $\omega_\delta -$ regular.

The normality and $\omega_\delta -$ normality are independent as shown in the following examples:

Example 5.17. Consider R with the usual topology. Then $(R, \mathfrak{A})$ is normal space, whenever it is not $\omega_\delta -$ normal, since $(R, \mathfrak{A})$ is $\omega_\delta - T_1$ but not $\omega_\delta -$ regular.

Example 5.18. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then (X, τ) is $\omega_\delta -$ normal, but not normal space because there is no disjoint open sets, which containing disjoint closed sets $\{b\}$ and $\{c\}$, respectively.

Theorem 5.19. A space X is $\omega_\delta -$ normal if and only if for every continuous function $f: (F, \omega_\delta O(X)_F) \to ([a, b], \tau_{[a,b]})$, where F is an $\omega_\delta -$ closed subset of X can be extended to an $\omega_\delta -$ continuous function.

Definition 5.20. A family $\{A_\lambda: \lambda \in \Lambda\}$ of subsets of a space (X, τ) is called $\omega_\delta -$ locally finite if for each $x \in X$, there exists an $\omega_\delta -$ open set G containing x such that $\{\lambda \in \Lambda: G \cap A_\lambda \neq \emptyset\}$ is finite.

Definition 5.21. A family $\{A_\lambda: \lambda \in \Lambda\}$ of subsets of a space (X, τ) is called $\omega_\delta -$ discrete if for each $x \in X$, there is an $\omega_\delta -$ open set G containing x such that $\{\lambda \in \Lambda: G \cap A_\lambda \neq \emptyset\}$ has at most one member.

The $\omega_\delta -$ locally finite and locally finite are independent in general, as shown in the following two examples:

Example 5.22. Consider the set N of natural number with indiscrete topology. Then the family $\{\{n\}: n \in N\}$ is $\omega_\delta -$ locally finite, but not locally finite.

Example 5.23. Consider the space (R, T_{COC}), then the set $A = \{\{n\}: n \in Z\}$ is locally finite, but not $\omega_\delta -$ locally finite.

Proposition 5.24. If $\{A_\lambda: \lambda \in \Lambda\}$ is an $\omega_\delta -$ locally finite family of subsets of X, then $\{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\}$ is also $\omega_\delta -$ locally finite and $\omega_\delta Cl(\cup \{A_\lambda: \lambda \in \Lambda\}) = \cup \{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\}$.

Proof. Let $x \in X$. Since, $\{A_\lambda: \lambda \in \Lambda\}$ is $\omega_\delta -$ locally finite, so there exists an $\omega_\delta -$ open set G containing x such that the set $\{\lambda \in \Lambda: G \cap A_\lambda \neq \emptyset\}$ is finite. Since, $G \cap A_\lambda = \emptyset$ if and only if $G \cap \omega_\delta Cl(A_\lambda) = \emptyset$, so $\{\lambda \in \Lambda: G \cap \omega_\delta Cl(A_\lambda) \neq \emptyset\}$ is finite. Hence, $\{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\}$ is $\omega_\delta -$ locally finite**.** Since, $\cup \{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\} \subseteq \omega_\delta Cl(\cup \{A_\lambda: \lambda \in \Lambda\})$. To prove $\omega_\delta Cl(\cup \{A_\lambda: \lambda \in \Lambda\}) \subseteq \cup \{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\}$.

Let $x \notin \cup \{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\}$. Since by what we have proved above $\{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\}$ is $\omega_\delta -$ locally finite, so there exist an $\omega_\delta -$ open set U containing x such that $\Lambda_0 = \{\lambda \in \Lambda: U \cap \omega_\delta Cl(A_\lambda) \neq \emptyset\}$ is finite. Set $V = U \cap (\cap \{X - \omega_\delta Cl(A_\lambda): \lambda \in \Lambda_0\})$ is $\omega_\delta -$ open subsets of X containing x such that $V \cap (\cup \{A_\lambda: \lambda \in \Lambda\}) = \cup \{V \cap A_\lambda: \lambda \in \Lambda\} = \emptyset$. Thus, $x \notin \omega_\delta Cl(\cup \{A_\lambda: \lambda \in \Lambda\})$. Thus, $\omega_\delta Cl(\cup \{A_\lambda: \lambda \in \Lambda\}) \subseteq \cup \{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\}$. Therefore, $\omega_\delta Cl(\cup \{A_\lambda: \lambda \in \Lambda\}) = \cup \{\omega_\delta Cl(A_\lambda): \lambda \in \Lambda\}$.

Definition 5.25. An $\omega_\delta -$ open covering $\{U_\lambda: \lambda \in \Lambda\}$ of a space X is said to be $\omega_\delta -$ shrinkable, if there exists an $\omega_\delta -$ open covering $\{V_\lambda: \lambda \in \Lambda\}$ of X such that $\omega_\delta Cl(V_\lambda) \subseteq U_\lambda$, for each $\lambda \in \Lambda$.

Theorem 5.26. Let X be a space. Then the following statements are equivalent:

1) X is $\omega_\delta -$ normal.

2) Each point finite $\omega_\delta -$ open covering of X is $\omega_\delta -$ shrinkable.

3) Each finite $\omega_\delta -$ open covering of X has an $\omega_\delta -$ locally finite $\omega_\delta -$ closed refinement.

Proof.

1 ⟹ 2: Let $\{U_\lambda: \lambda \in \Lambda\}$ be a point finite $\omega_\delta -$ open covering of an $\omega_\delta -$ normal space X. Assume that Λ is well – ordered. We shall construct the $\omega_\delta -$ shrinking to $\{U_\lambda: \lambda \in \Lambda\}$ by transfinite induction. For this, let μ be an element of Λ and suppose that for each $\lambda < \mu$, we have an $\omega_\delta -$ open set V_λ such that $\omega_\delta Cl(V_\lambda) \subseteq U_\lambda$ and for each $v < \mu$, $[\cup \{V_\lambda: \lambda < v\}] \cup [\cup \{U_\lambda: \lambda \geq v\}] = X$. Let $x \in X$. Since, $\{U_\lambda: \lambda \in \Lambda\}$ is point finite, so there is a largest element, say $\xi \in \Lambda$ such that $x \in U_\xi$. If $\xi \geq \mu$, then $x \in \cup \{U_\lambda: \lambda \geq \mu\}$. However, if $\xi < \mu$, then $x \in [\cup \{V_\lambda: \lambda < \mu\}]$. Hence, $[\cup \{V_\lambda: \lambda < \mu\}] \cup [\cup \{U_\lambda: \lambda \geq \mu\}] = X$. Thus, U_μ contains the complement of an $\omega_\delta -$ open set $[\cup \{V_\lambda: \lambda < \mu\}] \cup [\cup \{U_\lambda: \lambda > \mu\}]$. Since, X is an $\omega_\delta -$ normal space, so by Theorem 5.12, there exist an $\omega_\delta -$ open set, say V_μ such that $(X - [\cup \{V_\lambda: \lambda < \mu\}] \cup [\cup \{U_\lambda: \lambda > \mu\}]) \subseteq V_\mu \subseteq \omega_\delta Cl(V_\mu) \subseteq U_\mu$. Hence, $[\cup \{V_\lambda: \lambda \leq \mu\}] \cup [\cup \{U_\lambda: \lambda \geq \mu\}] = X$. Thus, the construction of $\omega_\delta -$ shrinking of $\{U_\lambda: \lambda \in \Lambda\}$ is completed by transfinite induction.

2 ⟹ 3: Obvious.

3 ⟹ 1: Let X be a space which satisfies condition (3), let U and V be two $\omega_\delta -$ open subsets of X such that $U \cup V = X$. Then $\{U, V\}$ is a finite $\omega_\delta -$ open covering of X. Then by hypothesis, this covering has $\omega_\delta -$ locally finite $\omega_\delta -$ closed refinement, say ψ. Let F and H be the union of these members of ψ which is contained in U and V, respectively. Then by Proposition 5.24, F and H are $\omega_\delta -$ closed subsets of X. Since, ψ is a cover of X, so X is $\omega_\delta -$ normal space.

Theorem 5.27. Let $\{G_\lambda : \lambda \in \Lambda\}$ be an $\omega_\delta -$ locally finite family of an $\omega_\delta -$ open sets of an $\omega_\delta -$ normal space X, and let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of $\omega_\delta -$ closed sets such that $E_\lambda \subseteq G_\lambda$, for each $\lambda \in \Lambda$. Then there exists a family $\{V_\lambda : \lambda \in \Lambda\}$ of $\omega_\delta -$ open sets such that $E_\lambda \subseteq V_\lambda \subseteq \omega_\delta Cl(V_\lambda) \subseteq G_\lambda$, for each $\lambda \in \Lambda$ and the families $\{E_\lambda : \lambda \in \Lambda\}$ and $\{\omega_\delta Cl(V_\lambda) : \lambda \in \Lambda\}$ are similar.

References:

[1] Alias B. Khalaf, Halgwrd M.D., and K. Kannan, "*Some Types of Separation Axioms in Topological Spaces*", Tamsui Oxford Journal of Information and Mathematical Sciences. Vol. 28, No. 3, pp. 303-326 Aletheia University. (2012).

[2] H. Hdeib," *$\omega -$ closed Mapping*", Revista Colombiana de Math. Vol. 16, No. 1-2, pp. 65-78. (1982).

[3] Halgwrd M. D., "*A new Topology from an old one*", Journal of the chungcheong mathematical society. Vol. 25, No. 3, (2012).

[4] I. L. Reilly, M. K. Vamanamurthy, "*On $\delta -$ Continuous Functions*", Indian J. Pure appl. Math. Vol. 15, No. 1, pp. 83-88. (1984).

[5] Jeffreys, H. and Jeffreys, B. S., "*Uniform Convergence of Sequences and Series*", Cambridge, England: Cambridge University Press, pp. 37-43, (1988).

[6] K. Y. Al-Zoubi and B. Al-Nashief, "*The Topology of $\omega -$ open subsets*", Al-Manarah. Vol 9, No. 2, pp. 169-179. (2003).

[7] M. H. Stone, "*Applications of the Theory if Boolean rings to General Topology*", Amer. Math. Soc. Trans. Vol. 41, pp. 375-381. (1937).

[8] N. V. Veli¸cko, "*H-closed Topological Spaces*", Amer. Math. Soc. Trans. Vol. 2, No. 78, pp. 103-118 (Mat. Sbornik Vol. 70, pp. 98-112). (1968).

[9] Pears A. R., "*Dimension Theory of General Spaces*", Cambridge Uinversity, Printed in Great Britain at the University Printing House, Cambridge, (1975).

[10] Sharma J. N., and Chauhan J. P., "*Topology (general and algebraic)*", Krishna Prakashsan Media, Indai. (2011).